

# Generating spherically symmetric static perfect fluid solutions

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## Abstract

By a choice of new variables the pressure isotropy condition for spherically symmetric static perfect fluid spacetimes can be made a quadratic algebraic equation in one of the two functions appearing in it. Using the other variable as a generating function, the pressure and the density of the fluid can be expressed algebraically by the function and its derivatives. One of the functions in the metric can also be expressed similarly, but to obtain the other function, related to the redshift factor, one has to perform an integral. Conditions on the generating function ensuring regularity and physicality near the center are investigated. Two everywhere physically well behaving example solutions are generated, one representing a compact fluid body with a zero pressure surface, the other an infinite sphere.

## 1 Introduction

The aim of this paper is to discuss an algorithm that can be used to generate any number of physically realistic pressure and density profiles for spherical perfect fluid distributions without calculating integrals. Our key step is the

transformation of the field equation into a form algebraic in one variable. A similar equation was already written down in the paper of Burlankov[1]. However, it was immediately transformed to isotropic coordinates, thereby obtaining a generating formalism similar to that of Glass and Goldman[2][3], where even the calculation of pressure and density requires integrals.

Because of their inability to support shear stresses static perfect fluid configurations are expected to be spherically symmetric. Theorems showing this have been proved under fairly general conditions[4][5]. In the static spherically symmetric case the only relation one gets from the Einstein equations is the pressure isotropy condition. This condition together with a fluid equation of state and a central density value can be used to determine a spherically symmetric regular matter distribution, which either connects to an exterior vacuum region through a zero pressure hypersurface or extends to the whole spacetime. The uniqueness of this spacetime has been proved by Rendall and Schmidt[6] for monotonic equation of states. Regularity of the metric can be shown even without the monotonicity assumption[7].

Since the addition of even the simplest equation of state makes the task of finding an exact solution extremely difficult, the usual method to proceed is to look for some solution of the pressure isotropy condition under some mathematical assumptions and hope that the resulting equation of state will be simple and physical. The most common way to choose the radial coordinate  $r$  is by setting the surface of the isometry spheres to  $4\pi r^2$ . Then by appropriately choosing the form of the two functions describing the metric, the pressure isotropy condition is a first order linear equation in one of the two variables. As already was pointed out by Wyman[8], for arbitrary choice of the other function this equation can always be solved by quadratures. In the other variable the pressure isotropy condition is either a second order linear differential equation or a first order Riccati equation, depending on the exact choice of the function. Considering this relative mathematical simplicity, it is not surprising that more than hundred static perfect fluid exact solutions have already appeared in the literature[9]. Unfortunately a large part of these solutions are unphysical in several aspects. Many do not have a regular center, others have no positive pressure and density, violate the dominant energy condition, or have unphysical sound speed.

Even though the field equation is in principle solvable by quadratures, the resulting integrals can be calculated in terms of elementary functions only in some exceptional cases. The imposition of physicality near the center makes the possible generating functions even more complicated and the chance of

being able to calculate the integrals even less. There are some techniques that can be used to generate new solutions from known ones published in the literature[10][11], but they also require calculation of integrals or solution of differential equations. Using isotropic coordinates, Kuchowicz[12] writes the pressure isotropy condition into a form which is algebraic in one of its two variables. Glass and Goldman[2][3] introduce pressure and density related variables, also in isotropic coordinates, to obtain a similar algebraic equation. They use one of the two variables as a generating function, and obtain formulae for the pressure, the density and the metric functions involving integrals of functional expressions of the generating function and its derivatives. They also give conditions on the generating function which ensure that the resulting metric is physically realistic near the origin. In the present paper we follow a similar approach in area coordinates  $r$ . The main advantage of our proposed method is that it does not require the calculation of any integrals for the expression of the pressure, the density and one of the metric functions. A similar equation to our main equation has been already calculated by Burlankov[1], but a transformed form of it was used there to generate solutions.

In Section 2 we transform the pressure isotropy condition into a quadratic algebraic equation in one of its variables and propose to use the other variable as a generating function. The pressure, the density and one of the functions in the metric are expressed as algebraic expressions of the generating function and its first and second derivatives. The other function, determining the redshift factor, appears only in differentiated form in the field equation, and consequently only its derivative can be given in an algebraic form. The functional form of the generating function is calculated for some known solutions in Section 3. A power series expansion near the center is used in Section 4 in order to establish necessary conditions on the generating function to make the center regular, the density and pressure positive, having a local maximum, and satisfying the dominant energy condition near the center. The metric induced by the simplest physically realistic polynomial generating function is calculated in Section 5. The solution represents a fluid sphere with a regular center and a zero pressure surface. It has physical density and pressure and casual sound speed for some choice of the parameters. In Section 6, a second simple choice of the generating function is used to calculate a further solution, for which the matter extends to infinity.

## 2 Field equations

The metric  $g_{\mu\nu}$  of a general stationary spherically symmetric configuration can be written in area coordinates as

$$ds^2 = -e^{2\nu} dt^2 + \frac{1}{B} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (1)$$

where  $\nu$  and  $B$  are functions of the radial coordinate  $r$ . We assume that the spacetime region is filled with perfect fluid, and the fluid velocity vector  $u^\mu$  is proportional to  $\partial/\partial t$ . Since solutions of the Einstein equations with nonzero cosmological constant can be interpreted as solutions with negative pressure or density contributions, we do not include a cosmological term explicitly in our equations. Calculating the Einstein equations  $G_{\mu\nu} = 8\pi T_{\mu\nu}$ , we obtain the energy density, the radial and the angular directional pressures as

$$\mu = \frac{1}{8\pi r^2} (1 - B - rB') , \quad (2)$$

$$p_r = \frac{1}{8\pi r^2} (2rB\nu' + B - 1) , \quad (3)$$

$$p_\vartheta = \frac{1}{16\pi r} [2rB\nu'' + 2r^2(\nu')^2 + r\nu'B' + 2B\nu' + B'] , \quad (4)$$

where the prime denotes derivatives with respect to the radial coordinate  $r$ . The only field equation one gets is the pressure isotropy condition  $p \equiv p_r = p_\vartheta$ , which takes the form

$$r(r\nu' + 1)B' + [2r^2\nu'' + 2r^2(\nu')^2 - 2r\nu' - 2]B + 2 = 0 . \quad (5)$$

This equation together with an equation of state  $p = p(\mu)$  and a central density  $\mu_c$  determine a unique spherically symmetric perfect fluid configuration, at least in the  $\frac{dp}{d\mu} > 0$  case[6]. In this paper, however, we try to find the general solution of the pressure isotropy condition  $p_r = p_\vartheta$  and leave the task of interpreting the resulting equation of states as a second step. Equation (5) is first order and linear in  $B$ , and consequently, for any given function  $\nu$ , the general solution for  $B$  can be given in terms of integrals[8]. We instead proceed by transforming (5) into a simpler form.

Because of the freedom in constant rescaling of the time coordinate  $t$ , only the derivatives of the function  $\nu$  appears in the expressions for  $\mu$ ,  $p_r$  and  $p_\vartheta$ . This would suggest to introduce  $\nu'$  as a new variable. However, considering

the coefficient of  $B'$  in (5), it turns out to be more useful to introduce a new function as

$$\beta = r\nu' + 1 . \quad (6)$$

Then the field equation takes the simple form

$$r\beta B' + 2rB\beta' + 2\beta^2 B - 8\beta B + 4B + 2 = 0 , \quad (7)$$

and the radial pressure becomes

$$p_r = \frac{1}{8\pi r^2} (2\beta B - B - 1) . \quad (8)$$

Introducing a further new variable

$$\alpha = \beta^2 B , \quad (9)$$

equation (7) becomes a second order algebraic equation in  $\beta$ ,

$$2(\alpha + 1)\beta^2 + (r\alpha' - 8\alpha)\beta + 4\alpha = 0 . \quad (10)$$

If instead of  $\alpha$  one introduces its square root,  $z = \sqrt{\alpha}$  as a new function, and denote  $\sqrt{B} = b$ , then it is possible to get from (7) an equation algebraic in  $b$ , which is already given in the paper of Burlankov[1],

$$2b^2 + (rz' - 4z)b + z^2 + 1 = 0 . \quad (11)$$

Although at first sight this equation appears as a more comfortable form directly giving the metric function  $b = \sqrt{B}$  from the generating function  $z = \sqrt{\alpha}$ , it has serious disadvantages. First of all, for most known perfect fluid solutions  $B$  and  $\alpha$  are not a square, and the appearance of the further square roots makes the calculations even more cumbersome. The more serious second problem is that the simplest polynomial choices for  $z$  do not appear to give appropriate results. For example, the quadratic equation (11) has no real solution for  $b$  when choosing  $z = 1 - ar^2$  with  $a$  constant, while  $\alpha = 1 - ar^2$  gives the Einstein static universe, as we will see in Section 3. In general, introducing any functional form of  $\alpha$ , such as  $\alpha^2$ ,  $\frac{1}{\alpha}$  or  $e^\alpha$ , as a new variable would give a different algebraic equation in the other variable. This makes our formalism non-unique in one hand, but gives opportunities to generate even more variety of new solutions on the other hand.

A further disadvantage of the equation (11) compared to (10) that it is nonlinear in both variables. Since equation (10) is linear in  $\alpha$ , if the function

$\nu$  and consequently  $\beta = r\nu' + 1$  is given, one can express its general solution for  $\alpha$  using integrals,

$$\alpha = e^{-I_\beta} \left( C - 2 \int \frac{\beta}{r} e^{I_\beta} dr \right), \quad (12)$$

$$I_\beta = \int \frac{2}{r\beta} (\beta^2 - 4\beta + 2) dr, \quad (13)$$

where  $C$  is some constant. Unfortunately, there is no guarantee, that for those  $\nu$  for which the integrals can be given in terms of elementary functions, the resulting equations of states will be also physically reasonable. Instead of trying to investigate this further we focus on the alternative approach, considering the function  $\alpha$  as the basic quantity.

For any function  $\alpha$  for which

$$(8\alpha - r\alpha')^2 > 32\alpha(\alpha + 1) \quad (14)$$

the quadratic equation (10) has two solutions for the function  $\beta$ . We denote them by  $\beta_+$  and  $\beta_-$ ,

$$\beta_\pm = \frac{1}{4(\alpha + 1)} \left[ 8\alpha - r\alpha' \pm \sqrt{(8\alpha - r\alpha')^2 - 32\alpha(\alpha + 1)} \right]. \quad (15)$$

The condition (14) will turn out to be not particularly restrictive, since as we will see in Section 4, at a regular center  $\alpha \rightarrow 1$  and then (14) holds as an equality at  $r = 0$ . Furthermore, we will see that positivity of the fluid density will ensure the existence of the square root near the center. Actually, it will also turn out that the root  $\beta_-$  always belong to non-positive densities, and hence it is unphysical. For actual calculations it may be convenient to use the identity

$$\beta_+ \beta_- = \frac{2\alpha}{\alpha + 1} \quad (16)$$

to eliminate square roots from the denominator.

Using (9) and (6), the metric functions  $B_+$  and  $\nu_+$  belonging to  $\beta_+$ , and  $B_-$  and  $\nu_-$  belonging to  $\beta_-$ , can be calculated as

$$B_\pm = \frac{\alpha}{\beta_\pm^2} = \frac{(\alpha + 1)^2}{4\alpha} \beta_\mp^2, \quad (17)$$

$$\nu_\pm = \int_0^r \frac{1}{r} (\beta_\pm - 1) dr + C_\pm, \quad (18)$$

where  $C_+$  and  $C_-$  are constants determining the scaling of the time coordinate  $t$ . The integral (18) generally cannot be given in terms of elementary functions. However, as can be seen from (2) and (8), the physically important pressure and density can be calculated without performing integrals. Denoting the pressure and density belonging to  $\beta_+$  by  $p_+$  and  $\mu_+$ , and those belonging to  $\beta_-$  by  $p_-$  and  $\mu_-$ ,

$$p_{\pm} = \frac{1}{8\pi r^2} (2\beta_{\pm} B_{\pm} - B_{\pm} - 1) , \quad (19)$$

$$\mu_{\pm} = \frac{1}{8\pi r^2} (1 - B_{\pm} - r B'_{\pm}) . \quad (20)$$

The form of the equation of state is extremely complicated in general, and possibly cannot even be put into an  $r$  independent  $\mu = \mu(p)$  or  $f(\mu, p) = 0$  form using elementary functions. However, the important point is that, in principle, all static spherically symmetric perfect fluid solutions, with all possible equations of states, could be given by suitably choosing the function  $\alpha$ . Indeed, for any known solution, first  $\beta$ , then  $\alpha$  can be calculated in terms of the metric functions  $\nu$  and  $B$ , using the equations (6) and (9).

### 3 Some known solutions

In this section we look at the form of some known exact solutions in our formalism. Since the field equation (10) has in general two roots, most of these metrics are paired with a perfect fluid counterpart solution.

The simplest example is the Minkowski spacetime with  $\alpha = 1$ . This is the only constant  $\alpha$  spacetime with a regular center. Then the quadratic equation (10) has only one solution,  $\beta = 1$ .

Next we consider the vacuum Schwarzschild spacetime with mass parameter  $m$ . Then  $e^{2\nu} = B = 1 - 2m/r$ , and from (6) and (9)

$$\beta = \frac{r - m}{r - 2m} , \quad (21)$$

$$\alpha = \frac{(m - r)^2}{r(r - 2m)} = -\frac{m}{2r} + \frac{3}{4} - \frac{r}{8m} - \frac{r^2}{16m^2} + O(r^3) . \quad (22)$$

However, given this form of  $\alpha$ , (21) is only one of the solutions of (10). The other solution, corresponding to the negative sign in (15) is

$$\beta_- = 2 \frac{(r - m)(r - 2m)}{m^2 - 4mr + 2r^2} . \quad (23)$$

This belongs to a perfect fluid spacetime with pressure and density

$$p_- = \frac{m^3(7m - 4r)}{32\pi r^3(r - 2m)^3} , \quad (24)$$

$$\mu_- = \frac{m^2(2r - 3m)(10r - 17m)}{32\pi r^2(r - 2m)^4} . \quad (25)$$

The solution has a complicated equation of state and does not have a regular center at  $r = 0$ .

The next simplest solution with a regular center is the Einstein static universe with constant  $\nu$ , with  $\beta = 1$  and  $\alpha = B = 1 - ar^2$ , where  $a$  is some positive constant. The other root of (10) is

$$\beta_- = 2 \frac{1 - ar^2}{2 - ar^2} . \quad (26)$$

Using (17) and (18), the functions in the metric (1) turn out to be

$$e^{2\nu_-} = C(2 - ar^2) , \quad B_- = \frac{(2 - ar^2)^2}{4(1 - ar^2)} . \quad (27)$$

The pressure

$$p_- = \frac{a(3ar^2 - 4)}{32\pi(1 - ar^2)} \quad (28)$$

can be positive for negative  $a$ , but the density

$$\mu_- = \frac{a^2 r^2 (3ar^2 - 5)}{32\pi(1 - ar^2)^2} \quad (29)$$

is always negative near the center and vanishes for  $r = 0$ . Although this solution is unphysical in the zero cosmological constant case, it has a regular center, and the fluid obeys a remarkably simple equation of state

$$8\pi a \mu_- = -(a + 8\pi p_-)(3a + 64\pi p_-) . \quad (30)$$

The interior Schwarzschild solution is described by the metric components

$$e^{2\nu} = b^2 \left( a - \sqrt{1 - \frac{r^2}{R^2}} \right)^2 , \quad B = 1 - \frac{r^2}{R^2} , \quad (31)$$



where  $a$ ,  $b$  and  $R$  are constants. The pressure is positive if  $1 < a < 3$ . This metric can be obtained in our formalism by choosing

$$\begin{aligned}\alpha &= \left[ \sqrt{1 - \frac{r^2}{R^2}} + \frac{r^2}{R^2 \left( a - \sqrt{1 - \frac{r^2}{R^2}} \right)} \right]^2 \\ &= 1 + \frac{3-a}{R^2(a-1)}r^2 - \frac{r^4}{R^4(a-1)} - \frac{a^2-3a+4}{4R^6(a-1)^3}r^6 + O(r^8)\end{aligned}\tag{32}$$

and taking the root

$$\beta_+ = 1 + \frac{r^2}{r^2 - R^2 + aR^2\sqrt{1 - \frac{r^2}{R^2}}}\tag{33}$$

of the quadratic equation (10). The other solution  $\beta_-$  gives another perfect fluid metric with a regular center and a complicated equation of state. Unfortunately, the matter density  $\mu_-$  turns out to be negative near the center when the pressure is positive. We will see in the next section that this is a general property. If one solution of (10) is physically well behaving, then the other belongs to negative matter densities.

## 4 Expansion around a regular center

In order for the metric (1) to possess a regular center the functions  $\nu$  and  $B$  must have regular limits at  $r = 0$ . To obtain the standard area per radius ratio for small spheres the limit of  $B$  must be 1. Further restrictions may come from differentiability conditions in a coordinate system which is regular at the center, and also from the regularity of the pressure and density. It can be seen from (6) and (9) that the limit of the functions  $\beta$  and  $\alpha$  at  $r = 0$  must be also 1.

We take the expansion of the function  $\alpha$  in the form

$$\alpha = \sum_{i=0}^{\infty} \alpha_i r^i, \tag{34}$$

where  $\alpha_i$  are constant expansion coefficients, and  $\alpha_0 = 1$ . If we assumed that  $\nu$  and  $B$ , and consequently  $\alpha$  and  $\beta$ , are analytic at the center in a rectangular coordinate system, then because of the spherical symmetry, all

their odd expansion coefficients would have to vanish, i. e.  $\alpha_i$  would have to be zero for odd  $i$ . We proceed without this assumption now, and examine the first few expansion coefficients one by one.

If  $\alpha_1$  is nonzero, calculating  $\beta$  from (15),  $B$  from (17),  $p$  and  $\mu$  from (19) and (20), the leading term in the pressures  $p_+$  and  $p_-$  turns out to be proportional to  $\frac{\alpha_1}{r}$ , and the leading term in the densities  $\mu_+$  and  $\mu_-$  is proportional to  $\frac{\sqrt{\alpha_1}}{r^{3/2}}$ . If  $\alpha_1 = 0$  then the pressure is regular but the densities are proportional to  $\frac{\sqrt{\alpha_3}}{\sqrt{r}}$ . This shows that  $\alpha_1$  and  $\alpha_3$  must be zero for regular matter distributions. No such restriction follows for  $\alpha_5$  and higher odd index coefficients.

When  $\alpha_1 = \alpha_3 = 0$  then the expression under the square root in (15) has the expansion

$$(8\alpha - r\alpha')^2 - 32\alpha(\alpha + 1) = 4(\alpha_2^2 - 8\alpha_4)r^4 - 48\alpha_5r^5 - 16(\alpha_2\alpha_4 + 4\alpha_6)r^6 + O(r^7) . \quad (35)$$

In order for  $\beta$  to exist, this must be non-negative, which holds near the center only if  $8\alpha_4 \leq \alpha_2^2$ . There are two different spacetimes belonging to a given  $\alpha$ . One belongs to the positive sign in (15), and the other to the negative sign. The expansions of the pressures belonging to  $\beta_+$  and  $\beta_-$  are

$$8\pi p_+ = \alpha_2 - \frac{1}{8} \left( \alpha_2^2 + \alpha_2 \sqrt{\alpha_2^2 - 8\alpha_4 - 12\alpha_4} \right) r^2 + \frac{\alpha_5}{4} \left( \frac{3\alpha_2}{\sqrt{\alpha_2^2 - 8\alpha_4}} + 7 \right) r^3 + O(r^4) , \quad (36)$$

$$8\pi p_- = \alpha_2 - \frac{1}{8} \left( \alpha_2^2 - \alpha_2 \sqrt{\alpha_2^2 - 8\alpha_4 - 12\alpha_4} \right) r^2 - \frac{\alpha_5}{4} \left( \frac{3\alpha_2}{\sqrt{\alpha_2^2 - 8\alpha_4}} - 7 \right) r^3 + O(r^4) . \quad (37)$$

These readily show that the pressure can be positive only if  $\alpha_2 > 0$ . The expansions of the fluid densities are

$$8\pi\mu_+ = \frac{3}{2} \left( \sqrt{\alpha_2^2 - 8\alpha_4} - \alpha_2 \right) - \frac{12\alpha_5}{\sqrt{\alpha_2^2 - 8\alpha_4}} r - \frac{5}{8} \left[ \alpha_2^2 - 4\alpha_4 + \frac{\alpha_2^3 + 32\alpha_6}{\sqrt{\alpha_2^2 - 8\alpha_4}} + \frac{72\alpha_5^2}{(\alpha_2^2 - 8\alpha_4)^{3/2}} \right] r^2 + O(r^3) , \quad (38)$$

$$\begin{aligned}
8\pi\mu_- &= -\frac{3}{2}\left(\sqrt{\alpha_2^2 - 8\alpha_4} + \alpha_2\right) + \frac{12\alpha_5}{\sqrt{\alpha_2^2 - 8\alpha_4}}r \\
&\quad - \frac{5}{8}\left[\alpha_2^2 - 4\alpha_4 - \frac{\alpha_2^3 + 32\alpha_6}{\sqrt{\alpha_2^2 - 8\alpha_4}} - \frac{72\alpha_5^2}{(\alpha_2^2 - 8\alpha_4)^{\frac{3}{2}}}\right]r^2 + O(r^3) .
\end{aligned} \tag{39}$$

Since the positivity of the pressure requires  $\alpha_2 > 0$ , the density  $\mu_-$ , belonging to the negative sign in (15), is necessarily negative near the center. This shows that if one does not consider spacetimes with a cosmological constant, the  $\beta_-$  root of (10) is always unphysical. However, the density  $\mu_+$ , belonging to  $\beta_+$ , is positive if  $\alpha_4 < 0$  and  $\alpha_2 > 0$ . In this case the  $8\alpha_4 \leq \alpha_2^2$  condition automatically holds, and hence the expression under the square root in (15) is always positive. The dominant energy condition  $\mu \geq p$  also holds at the center if  $\alpha_4 \leq -\frac{2}{9}\alpha_2^2$ .

In a physical situation one expects the pressure and density to have a maximum at the center. This certainly holds for  $p_+$  in a neighborhood of the center, since the coefficient of  $r^2$  in (36) is positive when  $\alpha_2 > 0$  and  $\alpha_4 < 0$ . If  $\alpha_5 > 0$  then  $\mu_+$  also have a local maximum at the center. In the more realistic case, when  $\alpha_5 = 0$ , the decreasing nature of  $\mu_+$  gives a condition on  $\alpha_6$ ,

$$32\alpha_6 > -(\alpha_2^2 - 4\alpha_4)\sqrt{\alpha_2^2 - 8\alpha_4} - \alpha_2^3 , \tag{40}$$

which always holds when  $\alpha_6$  is positive.

Functions  $\alpha$  with  $\alpha_5 > 0$  seem to be physically realistic, except that their equation of state for the fluid has some bad properties. If an equation of state exists in the form  $p \equiv p(\mu)$  then

$$\frac{dp}{dr} = \frac{dp}{d\mu} \frac{d\mu}{dr} \tag{41}$$

must hold. However, at the center  $\frac{dp}{dr} = 0$  but  $\frac{d\mu}{dr}$  is nonzero. This shows that  $\frac{dp}{d\mu}$  must be zero when  $\mu$  takes its central value  $\mu_c$ , which means zero sound speed there. It would be important to know whether a realistic monotone increasing equation of state  $p \equiv p(\mu)$  rules out all odd index coefficients in the expansion of  $\alpha$ .

## 5 A solution representing a compact fluid sphere

The simplest physically realistic polynomial choice for the generating function  $\alpha$  appears to be

$$\alpha = 1 + acr^2 - \frac{1}{8}c^2(1 - 2a^2)r^4, \quad (42)$$

where  $a$  and  $c$  are positive constants satisfying  $a^2 < \frac{1}{2}$ , or rather  $a^2 < \frac{9}{34}$  in order to comply with the dominant energy condition near the center. For the sake of simplifying the square roots appearing from (15) we introduce a new radial variable  $x$  defined by

$$r^2 = \frac{4(\sin x + a)}{c(1 - 2a^2)}. \quad (43)$$

Then the variable  $x$  is restricted by  $x \geq x_c = \arcsin(-a)$ . The generating function  $\alpha$  takes the form

$$\alpha = \frac{1 - 2\sin^2 x}{1 - 2a^2}. \quad (44)$$

It still contains the parameter  $c$  through

$$\sin x = \frac{1}{4}c(1 - 2a^2)r^2 - a. \quad (45)$$

Using that now

$$r \frac{d}{dr} = \frac{2}{\cos x}(\sin x + a) \frac{d}{dx} \quad (46)$$

the function under the square root in (15) becomes

$$(8\alpha - r\alpha')^2 - 32\alpha(\alpha + 1) = \left(8 \cos x \frac{\sin x + a}{1 - 2a^2}\right)^2. \quad (47)$$

The root of (10) belonging to positive densities takes the simple form

$$\beta_+ = \frac{\sin x + \cos x}{\cos x - a}. \quad (48)$$

Using equations (17) and (18), one of the functions in the metric is

$$B_+ = \frac{(\cos x - a)^2 \cos(2x)}{(1 - 2a^2)[1 + \sin(2x)]}, \quad (49)$$

while the derivative of the other simplifies to

$$\frac{d\nu_+}{dx} = \frac{\cos x}{2(\cos x - a)} . \quad (50)$$

This can be integrated to yield

$$\nu_+ = \frac{x}{2} + \frac{a}{\sqrt{1-a^2}} \operatorname{arc tanh} \left( \frac{1+a}{\sqrt{1-a^2}} \tan \frac{x}{2} \right) \quad (51)$$

where an integration constant has been absorbed into the scaling of the time coordinate  $t$ .

Using  $x$  as the radial coordinate the metric takes the form

$$\begin{aligned} ds^2 = & -\exp \left[ x + \frac{2a}{\sqrt{1-a^2}} \operatorname{arc tanh} \left( \frac{1+a}{\sqrt{1-a^2}} \tan \frac{x}{2} \right) \right] dt^2 \\ & + \frac{[1 + \sin(2x)] \cos^2 x}{c \cos(2x)(\sin x + a)(\cos x - a)^2} dx^2 \\ & + \frac{4(\sin x + a)}{c(1-2a^2)} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \end{aligned} \quad (52)$$

This metric appears to be a new solution. Since the scale of the time coordinate  $t$  is arbitrary, the constant  $c$  corresponds to a constant conformal transformation of the metric. The pressure and density are

$$8\pi p = c \frac{(a - 3 \sin x) \cos x + (3a - \sin x) \sin x}{4(\sin x + \cos x)} \quad (53)$$

$$\begin{aligned} 8\pi\mu = & c \frac{7 \sin x - 11a + 2a \sin^2 x - 10 \sin^3 x}{4(\sin x + \cos x)^2} \\ & + c \frac{3 + 2a^2 - 4a \sin x - 3 \sin^2 x + 6a \sin^3 x}{2 \cos x (\sin x + \cos x)^2} . \end{aligned} \quad (54)$$

The fluid pressure and density are monotonically decreasing out from a regular center to a  $p = 0$  surface. The dominant energy condition  $\mu > p$  is satisfied everywhere for  $a < \sqrt{\frac{9}{34}} \approx 0.514$ , and the sound speed  $\frac{dp}{d\mu}$  is positive and less than one for  $a < 0.184$ . Unfortunately it is very difficult to express the equation of state in  $\mu = \mu(p)$  form, or even in an  $f(\mu, p) = 0$  form. The simplest expression the author could obtain is a complicated polynomial, eighth order in  $p$ ,  $a$  and  $c$ , and fourth order in  $\mu$ .

## 6 An infinite gaseous sphere solution

A further simple assumption on the generating function  $\alpha$  is that it is the ratio of two polynomials of the radial coordinate  $r$ . Considering the results in Section 4, the lowest degree form which might give a physically interesting solution is

$$\alpha = 1 + \frac{a^2 r^2}{1 + br^2} = 1 + a^2 (r^2 - br^4 + b^2 r^6 - b^3 r^8) + O(r^{10}) , \quad (55)$$

where  $a$  and  $b$  are positive constants. It is convenient to introduce a further constant  $c$  defined by

$$c^2 = \frac{2}{a^2}(b - a^2) \quad (56)$$

and use it in place of the constant  $b$ . The assumption that  $c$  is real and non-negative restricts the original constants into the range  $b \geq a^2$ . In order to simplify the square roots appearing from (15) while expressing  $\beta$ , we introduce a new radial variable  $x$  defined by

$$r^2 = \frac{2c - 3 \sinh x}{a^2 (2 + c^2) (2 \sinh x - c)} . \quad (57)$$

Then the center is at  $x_c = \text{arcsinh } \frac{2c}{3}$ , spatial infinity is at  $x_\infty = \text{arcsinh } \frac{c}{2}$ , and the new variable is restricted by  $0 < x_\infty \leq x \leq x_c$ . The  $b \leq a^2$  case could be treated in a similar way introducing sinus functions instead of sinus hyperbolics, but would lead to solutions which fail to satisfy the dominant energy condition  $\mu \geq p$  at infinity. The generating function  $\alpha$  takes the form

$$\alpha = \frac{4c + (c^2 - 4) \sinh x}{(2 + c^2) \sinh x} . \quad (58)$$

The function under the square root in (15) becomes a square

$$(8\alpha - r\alpha')^2 - 32\alpha(\alpha + 1) = \left( \frac{8c(2c - 3 \sinh x) \cosh x}{(2 + c^2) \sinh^2 x} \right)^2 . \quad (59)$$

The root of (10) belonging to positive densities takes the form

$$\beta_+ = \frac{(c \coth \frac{x}{2} - 2) \cosh \frac{x}{2}}{\cosh \frac{x}{2} + c \sinh \frac{x}{2}} . \quad (60)$$

Using equation (17), one of the functions in the metric is

$$B_+ = \frac{[4c + (c^2 - 4) \sinh x] \left(c \tanh \frac{x}{2} + 1\right)^2}{(2 + c^2) \left(c \coth \frac{x}{2} - 2\right)^2 \sinh x}, \quad (61)$$

while, from (18), the derivative of the other function becomes

$$\frac{d\nu_+}{dx} = \frac{c \cosh x}{4 \sinh \frac{x}{2} \left(\cosh \frac{x}{2} + c \sinh \frac{x}{2}\right) (c - 2 \sinh x)}. \quad (62)$$

This can be integrated to obtain

$$\begin{aligned} \nu_+ = & \frac{1}{2} \ln \sinh \frac{x}{2} + \frac{1}{2(3 + c^2)} \left[ 2\sqrt{4 + c^2} \operatorname{arctanh} \left( \frac{2 + c \tanh \frac{x}{2}}{\sqrt{4 + c^2}} \right) \right. \\ & \left. - (1 + c^2) \ln \left( \cosh \frac{x}{2} + c \sinh \frac{x}{2} \right) - \ln (2 \sinh x - c) \right]. \end{aligned} \quad (63)$$

Using  $x$  as the radial coordinate the metric takes the form

$$\begin{aligned} ds^2 = & -\exp(2\nu_+) dt^2 + \frac{2c - 3 \sinh x}{a^2 (2 + c^2) (2 \sinh x - c)} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \\ & + \frac{c^2 \cosh^2 x}{4a^2 (2 + c^2) (2c - 3 \sinh x) (2 \sinh x - c)^3 B_+} dx^2, \end{aligned} \quad (64)$$

where  $\nu_+$  is determined by equation (63) and  $B_+$  by (61). This second metric also appears to be a new solution. The constant  $a$  corresponds to a constant conformal transformation of the metric. The pressure and density are

$$8\pi p = a^2 \frac{(2 \sinh x - c)(2 - 2c^2 + 2 \cosh x + 5c \sinh x)}{4(c \cosh \frac{x}{2} - 2 \sinh \frac{x}{2}) \cosh^3 \frac{x}{2}}, \quad (65)$$

$$\begin{aligned} 8\pi\mu = & \frac{a^2(2 \sinh x - c)}{32 \left(c \cosh \frac{x}{2} - 2 \sinh \frac{x}{2}\right)^2 \cosh^4 \frac{x}{2} \cosh x} \left[ 3(7c^2 - 12) \sinh(3x) \right. \\ & + 6c(3c^2 - 4) \cosh(3x) + 2(12 + 61c^2 - 8c^4) \sinh(2x) \\ & + 4c(62 - 13c^2) \cosh(2x) + (156 - 475c^2 + 32c^4) \sinh x \\ & \left. + 2c(4 - 53c^2) \cosh x + 12c(13c^2 - 22) \right]. \end{aligned} \quad (66)$$

As indicated by numerical plots of the quantities, the fluid pressure and density are monotonically decreasing to zero, out from a regular center to

infinity. For any choice of the parameters  $a$  and  $c$ , the dominant energy condition  $\mu > p$  is satisfied, and the sound speed  $\frac{dp}{d\mu}$  is positive and less than one. Eliminating the variable  $x$ , one can obtain a complicated polynomial equation of state in the form  $f(\mu, p) = 0$ , eighth order in  $p$ , fourth order in  $\mu$ , and twelfth order in  $a$  and  $c$ . Near spatial infinity, in the small  $\mu$  and  $p$  limit, the equation of state is approximately linear,

$$\frac{p}{\mu} = \frac{6 + 4\sqrt{4 + c^2}}{2(4c^2 + 7)} < 1. \quad (67)$$

## 7 Conclusions

An algorithm has been given, which can be used to generate physically realistic density and pressure distributions from a generating function  $\alpha$  without calculating integrals. Any function  $\alpha$  of which the first few expansion coefficients satisfy the simple conditions stated in Section 4 generate spacetimes which are physically well behaving at least in a neighborhood of the center. The resulting pressure and density distributions can contain arbitrarily many parameters, for example by choosing  $\alpha$  to be a high order polynomial. Unfortunately, the actual forms of  $\mu$  and  $p$  can be quite complicated because of the square roots appearing. This makes the task of putting the resulting equation of state in a closed form very difficult. Unfortunately, a prescribed equation of state would yield a very complicated differential equation on the generating function. However, by trying many different functional forms for  $\alpha$ , hopefully one could find configurations with simple and physical equations of states.

The metric function  $\nu$  only appears in a differentiated form in the field equation and in the density and pressure expressions. Correspondingly, only its derivative  $\nu'$  can be given as an algebraic expression of  $\alpha$  and its first and second derivatives. Because of this, in some sense, one could consider a generated solution as an "exact solution" even if only the derivative of  $\nu$  can be given in a closed form. However, for some simpler choices of  $\alpha$ , the integral determining  $\nu$  can be calculated. This has been the case for the two example solution solutions presented in this paper.

It would be important to find out whether the generating formalism could be made simpler, or the obtainable exact solutions and equations of states could become more physical, for example, by choosing a functional expression



$f(\alpha)$  of the generating function instead of  $\alpha$ , or by using a different radial coordinate in place of the area coordinate  $r$ .

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## References

- [1] D. E. Burlankov, *Theor. and Math. Phys.* **95**, 455 (1993)
- [2] E. N. Glass and S. P. Goldman, *J. Math. Phys.* **19**, 856 (1978)
- [3] S. P. Goldman, *Astrophys J.* **226**, 1079 (1978)
- [4] R. Beig and W. Simon, *Commun. Math. Phys.* **144**, 373 (1992)
- [5] L. Lindblom and A. K. M. Masood-ul-Alam, *Commun. Math. Phys.* **162**, 123 (1994)
- [6] A. D. Rendall and B. G. Schmidt, *Class. Quantum Grav.* **8**, 985 (1991)
- [7] T. W. Baumgarte and A. D. Rendall, *Class. Quantum Grav.* **10**, 327 (1993)
- [8] M. Wyman, *Phys. Rev.* **75**, 1930 (1949)
- [9] M. S. R. Delgaty and K. Lake, *Computer Physics Comm.* **115**, 395 (1998)
- [10] H. Heintzmann, *Z. Phys.* **228**, 489 (1969)
- [11] P. G. Whitman, *Phys. Rev. D* **27**, 1722 (1983)
- [12] B. Kuchowicz, *Phys. Lett.* **35A**, 223 (1971)